RESEARCH PAPERS

Relations among gauge functions, metrics and Hausdorff measures*

WEN Shengyou[1] and WEN Zhiying[2]**

(1. Nonlinear Scientific Center, Wuhan University, Wuhan 430072, China; 2. Department of Mathematics, Tsinghua University, Beijing 100084, China)

Received August 28, 2002; revised October 28, 2002

Abstract In this paper we discuss the relations among the doubling condition, equivalence of the Hausdorff measures and equivalence of metrics. We will show that $\mathcal{H}^{\rho,g_1}$ and $\mathcal{H}^{\rho,g_2}$ are equivalent for any compact metric space (X,ρ) if and only if g_1 and g_2 are equivalent gauge functions. Then, we prove that for given $c\in(0,\infty)\setminus\{1\}$, $\mathcal{H}^{\rho,g}$ and $\mathcal{H}^{c\rho,g}$ are equivalent for any compact metric space (X,ρ) if and only if the gauge function g satisfies the doubling condition, where $\mathcal{H}^{\rho,g}$ is the Hausdorff measure with respect to the metric ρ and gauge function g.

Keywords: Hausdorff measure, metrics, gauge function, doubling condition.

Let X be a nonempty set. Two metrics ρ_1 and ρ_2 on X are said to be equivalent if there are constants $0<c_1,\ c_2<\infty$ such that for any $x,y\in X$

$$c_1\rho_1(x,y)\leqslant\rho_2(x,y)\leqslant c_2\rho_1(x,y).$$

Two measures μ_1 and μ_2 on X are said to be equivalent if there are constants $0<c_1,\ c_2<\infty$ such that for any $K\subset X$

$$c_1\mu_1(K)\leqslant\mu_2(K)\leqslant c_2\mu_1(K).$$

A function $g:[0,\infty)\to[0,\infty)$ is called a gauge function if g is continuous on the right for $t\geqslant 0$, monotonic increasing for $t\geqslant 0$, and $g(t)>0$ for $t>0$ with $g(0)=0$. Two gauge functions g_1 and g_2 are said to be equivalent if there are constants $0<c_1$, $c_2<\infty$ and $\delta>0$ such that for any $0\leqslant t\leqslant\delta$

$$c_1g_1(t)\leqslant g_2(t)\leqslant c_2g_1(t).$$

A gauge function g is called to satisfy the doubling condition if there are constants $0<c<\infty$ and $\delta>0$ such that for any $0\leqslant t\leqslant\delta$

$$g(2t)\leqslant cg(t).$$

Doubling condition plays an important role in the studies of geometric measure theory and fractal geometry, for example, see Mattila[1]. From above definitions, we see immediately a gauge function g satisfies the doubling condition if and only if $g(t)$ and $g(2t)$ are equivalent.

Let (X,ρ) be a metric space and g a gauge function and let $K\subset X$. A countable family $\{U_i\}$ of subsets of X is said to be a δ-covering of K with respect to the metric ρ if $K\subset\bigcup_i U_i$ with $0<|U_i|_\rho\leqslant\delta$ for each i, where $|U_i|_\rho$ denotes the diameter of U_i (that is, $|U_i|_\rho=\sup\{\rho(x,y);\ x,y\in U_i\}$). Set

$$\mathcal{H}_\delta^{\rho,g}(K)=\inf\sum_{i=1}^{\infty}g(|U_i|_\rho),$$

where the infimum is taken over all δ-coverings $\{U_i\}$ of K with respect to ρ. Then the Hausdorff measure with respect to the metric ρ and the gauge function g is defined by

$$\mathcal{H}^{\rho,g}(K)=\lim_{\delta\to 0}\mathcal{H}_\delta^{\rho,g}(K),$$

which is a Borel regular metric measure on X.

In this paper we will discuss the relations among the doubling condition, equivalence of the Hausdorff measures and equivalence of metrics. We will show that $\mathcal{H}^{\rho,g_1}$ and $\mathcal{H}^{\rho,g_2}$ are equivalent for any compact metric space (X,ρ) if and only if g_1 and g_2 are equivalent gauge functions (Theorem 1). Then, we prove that for given $c\in(0,\infty)\setminus\{1\}$, $\mathcal{H}^{\rho,g}$ and $\mathcal{H}^{c\rho,g}$ are equivalent for any compact metric space (X,ρ) if and only if the gauge function g satisfies the doubling condition (Theorem 2). Furthermore, we will show an extreme case if the gauge g does not

* Supported by the Major State Basic Research Development Program of China (Grant No. G2000077307)
** To whom correspondence should be addressed. E-mail: wenzy@tsinghua.edu.cn

fulfill the doubling condition, in this case, we will give a gauge function g for which there is a metric space (X, ρ) such that $0<\mathscr{H}^{\rho,g}(X)<\infty$, $\mathscr{H}^{c\rho,g}(X)=0$ for all $0<c<1$ and $\mathscr{H}^{c\rho,g}(X)=\infty$ for all $c>1$ (Proposition 1). Another aim of this paper is to study how the metric influences the Hausdorff measure. By the definition, if $g(t)=t^s$ with $s>0$, then the Hausdorff measure $\mathscr{H}^{\rho,g}$ scales with exponent s in the sense of $\mathscr{H}^{c\rho,g}=c^s\mathscr{H}^{\rho,g}$ for every $c>0$. Mauldin and Williams[2], Csömyei and Mauldin[3] investigated the scaling Hausdorff measures in the Eucildean spaces. They proved that, for every continuous gauge function g such that $\frac{g(t)}{t^d}$ is a decreasing function of t, the corresponding Hausdorff measure on $\mathbb{R}^d$ scales with exponent $0\leqslant s\leqslant d$ if and only if it is of the form $g(t)=t^sL(t)$, where $L(t)$ is slowly varying. By contrast, we show that even if the gauge g satisfies the doubling condition, there exists a metric space (X, ρ) with two equivalent metrics $c_1\rho$ and $c_2\rho$ $(c_1\neq c_2)$ such that $0<\mathscr{H}^{c_1\rho,g}(X)=\mathscr{H}^{c_2\rho,g}(X)<\infty$ (Proposition 2).

For every gauge function g we define

$$g_*(x)=\liminf_{t\to 0}\frac{g(tx)}{g(t)},$$

$$g^*(x)=\limsup_{t\to 0}\frac{g(tx)}{g(t)},\quad x\geqslant 0.$$

Given a gauge g, the following lemma gives some equivalent statements of the doubling condition of g related to g^* and g_*.

Lemma 1. Let g be a gauge function. The following statements are equivalent.

(i) g satisfies the doubling condition;

(ii) $g_*(x)>0$ for some $x\in(0,1)$;

(iii) $g_*(x)>0$ for all $x>0$;

(iv) $g^*(x)<\infty$ for some $x>1$;

(v) $g^*(x)<\infty$ for all $x>0$.

Proof. By the definitions of doubling condition and g_*, it is ready to see that g satisfies the doubling condition if and only if $g_*\left(\frac{1}{2}\right)>0$, and we obtain thus (i)⇒(ii) and (iii)⇒(i). Also, it is easily seen that $g_*(x)g^*\left(\frac{1}{x}\right)=1$ provided this product is not 0 times ∞. Thus (ii)⇔(iv) and (iii)⇔(v). To complete the proof, it suffices to show (ii)⇒(iii). For this, suppose $a\in(0,1)$ with $g_*(a)>0$. Then $g(at)>\frac{1}{2}g_*(a)g(t)$ if t is small enough. For any $x>0$ we may choose a positive integer m such that $x\geqslant a^m$, then $g(xt)\geqslant g(a^m t)\geqslant\left(\frac{1}{2}g_*(a)\right)^m g(t)$ for t small enough, this gives immediately $g_*(x)\geqslant\left(\frac{1}{2}g_*(a)\right)^m>0$.

The theorem below establishes the relation between the equivalence of Hausdorff measures and the equivalence of the gauges under the same metric.

Theorem 1. Let g, h be two gauges. Then $\mathscr{H}^{\rho,g}$ and $\mathscr{H}^{\rho,h}$ are equivalent for any compact metric space (X,ρ) if and only if g and h are equivalent.

Proof. The sufficiency of the condition is obvious.

Let $\frac{1}{2}<\lambda<1$ and $a_n=\lambda^{2^{-n}}$ $(n\in\mathbb{N})$, then $a_1a_2\cdots a_n>\lambda$ for any $n\geqslant 1$. Assume that g and h are not equivalent, then by the definition, there exists a sequence $\{\delta_n\downarrow 0\}_{n\geqslant 0}$ such that either $\lim_{n\to\infty}\frac{h(\delta_n)}{g(\delta_n)}=0$ or $\lim_{n\to\infty}\frac{h(\delta_n)}{g(\delta_n)}=+\infty$. We only discuss the case $\lim_{n\to\infty}\frac{h(\delta_n)}{g(\delta_n)}=0$. The case $\lim_{n\to\infty}\frac{h(\delta_n)}{g(\delta_n)}=+\infty$ can be treated in the same way. Since $\lim_{n\to\infty}g(\delta_n)=0$ from $g(0)=0$ and g being continuous on the right, we may suppose further the sequence $\{\delta_n\}$ is chosen to satisfy

$$g(\delta_n)\leqslant(1-a_n)g(\delta_{n-1}),\quad n\in\mathbb{N}.$$

We are going to construct a compact metric space (X,ρ) such that

$$0<\mathscr{H}^{\rho,g}(X)<\infty\quad\text{and}\quad\mathscr{H}^{\rho,h}(X)=0,$$

which follows the necessity of the theorem.

Take $k_n=\left[\frac{g(\delta_{n-1})}{g(\delta_n)}\right]$, $n\in\mathbb{N}$, where $[x]$ denotes the integer part of x, then

$$k_n\geqslant\left[\frac{1}{1-a_n}\right]\geqslant 2,$$

$$k_1k_2\cdots k_n\leqslant\frac{g(\delta_0)}{g(\delta_n)},\tag{1}$$

and

$$k_1k_2\cdots k_n\geqslant\left(\frac{g(\delta_0)}{g(\delta_1)}-1\right)\left(\frac{g(\delta_1)}{g(\delta_2)}-1\right)\cdots$$

$$\cdot\left(\frac{g(\delta_{n-1})}{g(\delta_n)}-1\right)$$
$$\geqslant\frac{a_1a_2\cdots a_n g(\delta_0)}{g(\delta_n)}\geqslant\frac{\lambda g(\delta_0)}{g(\delta_n)}. \qquad (2)$$

Let $F_0=[0,1]$. We construct a compact subset X of the interval $[0,1]$ in the following way: take any k_1 disjoint closed subintervals of the unit interval $[0,1]$ with positive lengths, and denote by F_1 the union of these k_1 intervals. For every element I of F_1, take k_2 disjoint closed subintervals of I with positive lengths, we obtain thus k_1k_2 disjoint closed intervals of $[0,1]$, and denote by F_2 the union of these k_1k_2 intervals. Continue the above procedure, we obtain a sequence $F_0\supset F_1\supset F_2\supset\cdots\supset F_n\cdots$. Set

$$X=\bigcap_{n=1}^{\infty}F_n.$$

By the above construction, X is a nonempty compact subset of $[0,1]$. Every element of F_n is called a basic interval of level-n. Denote by d_n the largest length of the basic intervals of level-n, we may require $\lim_{n\to\infty}d_n=0$.

Let $x,y\in X$ with $x\neq y$ and let $n(x,y)$ be the highest level of the basic interval which contains x and y, this means, there exists an interval I of level $n(x,y)$ which contains both x and y, but any basic interval does not contain simultaneously x, y if its level is higher than $n(x,y)$. We now define another metric ρ on X as follows:

$$\rho(x,y)=\begin{cases}0 & \text{if } x=y,\\ \delta_{n(x,y)} & \text{if } x\neq y.\end{cases}$$

Under this metric, we see easily that X is compact and totally disconnected.

We are going to estimate the Hausdorff measure of (X,ρ) with respect to the gauge g. Let $n\geqslant1$ and I a basic interval of level-n. From the definition of the metric ρ, we see that for any $x,y\in I$, $n(x,y)\geqslant n$, in which the equality holds for some $x,y\in I$, so $|I|_\rho=\delta_n$. Thus the family of all basic intervals of level-n is a δ_n-cover of X with respect to ρ, we have therefore from (1)

$$\mathscr{H}_{\delta_n}^{\rho,g}(X)\leqslant k_1k_2\cdots k_n g(\delta_n)\leqslant g(\delta_0),$$

which follows that

$$\mathscr{H}^{\rho,g}(X)\leqslant g(\delta_0). \qquad (3)$$

Let μ be the unique Borel probability measure on X satisfying

$$\mu(I_n)=\frac{1}{k_1k_2\cdots k_n},$$

where $n\geqslant1$ and I_n is any basic interval of level-n. Let U be a subset of X with $0<|U|_\rho<\delta_0$ and n the positive integer with $\delta_n\leqslant|U|_\rho<\delta_{n-1}$. By the definition of the metric ρ, we have $|U|_\rho=\delta_n$, so there is a basic interval of level-n I_n such that $U\subset I_n$. Thus we have from (2)

$$\mu(U)\leqslant\mu(I_n)=\frac{1}{k_1k_2\cdots k_n}\leqslant\frac{g(|U|_\rho)}{\lambda g(\delta_0)},$$

which yields from Frostman Lemma

$$\lambda g(\delta_0)\leqslant\mathscr{H}^{\rho,g}(X).$$

Associating with (3), we see that

$$0<\lambda g(\delta_0)\leqslant\mathscr{H}^{\rho,g}(X)\leqslant g(\delta_0)<\infty. \qquad (4)$$

With respect to the gauge h, we get also

$$\mathscr{H}_{\delta_n}^{h}(X)\leqslant k_1k_2\cdots k_n h(\delta_n)\leqslant g(\delta_0)\frac{h(\delta_n)}{g(\delta_n)}.$$

By the hypotheses $h(\delta_n)/g(\delta_n)\to0$, we get thus $\mathscr{H}^{\rho,h}(X)=0$. By comparing with (4), we see that $\mathscr{H}^{\rho,g}(X)$ and $\mathscr{H}^{\rho,h}(X)$ are not equivalent which completes the proof of the theorem.

For the same gauge function, the following theorem establishes the relation between the doubling condition and the equivalence of the Hausdorff measures.

Theorem 2. Suppose that g is a gauge and $c\in(0,\infty)\setminus\{1\}$. Then the following statements are equivalent.

(i) g satisfies the doubling condition;

(ii) $\mathscr{H}^{\rho,g}$ and $\mathscr{H}^{c\rho,g}$ are equivalent for any compact metric space (X,ρ).

Proof. (i)$\Rightarrow$(ii). In fact, we will prove a stronger implication: if g is a doubling gauge, then for any two equivalent metrics ρ_1 and ρ_2, $\mathscr{H}^{\rho_1,g}$ and $\mathscr{H}^{\rho_2,g}$ are equivalent.

Let $0<c_1$, $c_2<\infty$ be two constants such that

$$c_1\rho_1(x,y)\leqslant\rho_2(x,y)\leqslant c_2\rho_1(x,y),\quad\forall x,y\in X.$$

As g satisfies the doubling condition, $g_*(c_1)$ and $g^*(c_2)$ are finite positive by Lemma 1. Thus it suffices to prove for every $K\subset X$,

$$g_*(c_1)\mathscr{H}^{\rho_1,g}(K)\leqslant\mathscr{H}^{\rho_2,g}(K)\leqslant g^*(c_2)\mathscr{H}^{\rho_1,g}(K).$$

Let $K\subset X$ and $\delta>0$. Let $\{U_i\}$ be a δ-covering of K with respect to ρ_1, then $\{U_i\}$ is a $c_2\delta$-covering of K with respect to ρ_2. Since

$$\sum g(|U_i|_{\rho_2}) \leqslant \sum g(c_2|U_i|_{\rho_1}) \leqslant \sup_{0<t\leqslant\delta}\frac{g(c_2t)}{g(t)}\sum g(|U_i|_{\rho_1}),$$

we have

$$\mathscr{H}^{\rho_2,g}_{c_2\delta}(K) \leqslant \sup_{0<t\leqslant\delta}\frac{g(c_2t)}{g(t)}\mathscr{H}^{\rho_1,g}_{\delta}(K),$$

which yields that

$$\mathscr{H}^{\rho_2,g}(K) \leqslant g^*(c_2)\mathscr{H}^{\rho_1,g}(K). \quad (5)$$

In the same way, we have

$$\mathscr{H}^{\rho_1,g}(K) \leqslant g^*\left(\frac{1}{c_1}\right)\mathscr{H}^{\rho_2,g}(K).$$

As $g_*(c_1)g^*\left(\frac{1}{c_1}\right)=1$, we obtain

$$g_*(c_1)\mathscr{H}^{\rho_1,g}(K) \leqslant \mathscr{H}^{\rho_2,g}(K). \quad (6)$$

So $\mathscr{H}^{\rho_1,g}$ and $\mathscr{H}^{\rho_2,g}$ are equivalent from (5) and (6).

(ii)⇒(i). By the definition of Hausdorff measure, we see easily $\mathscr{H}^{c\rho,g}=\mathscr{H}^{\rho,g(c\cdot)}$. Thus, from Theorem 1, $g(t)$ and $g(ct)$ are equivalent, which follows by Lemma 1 that g satisfies the doubling condition.

The following proposition indicates an extreme case if the gauge g does not satisfy the doubling condition.

Proposition 1. Let $0<c<1$, let g be a gauge function and (X,ρ) a metric space. Suppose that

(i) $\lim_{t\to 0}\frac{g(ct)}{g(t)}=0$;

(ii) $0<\mathscr{H}^{\rho,g}(X)<\infty$.

Then

$$\mathscr{H}^{c\rho,g}(X)=0, \quad \mathscr{H}^{\frac{1}{c}\rho,g}(X)=\infty.$$

Proof. Note first there exist gauge function g and metric space (X,ρ) fulfilling the requirements of the proposition. For example, take

$$g(t)=\begin{cases}0 & \text{for } t=0,\\ 2^{-n-1} & \text{for } \frac{1}{n+1}\leqslant t<\frac{1}{n},\ n\in\mathbb{N}.\end{cases}$$

It is easily checked that $\lim_{t\to 0}\frac{g(ct)}{g(t)}=0$ for all $0<c<1$. Then, by a theorem of A. Dvoretzky (see [4], Theorem 36), there exists a metric space (X,ρ) such that $0<\mathscr{H}^{\rho,g}(X)<\infty$.

Now note that for each $\delta>0$, $\{U_i\}_{i=1}^{\infty}$ is a δ-cover of X with respect to ρ if and only if $\{U_i\}_{i=1}^{\infty}$ is a $c\delta$-cover of X with respect to $c\rho$, so by

$$\sum_{i=1}^{\infty}g(|U_i|_{c\rho})=\sum_{i=1}^{\infty}\frac{g(c|U_i|_{\rho})}{g(|U_i|_{\rho})}g(|U_i|_{\rho}) \leqslant \sup_{0<t\leqslant\delta}\frac{g(ct)}{g(t)}\sum_{i=1}^{\infty}g(|U_i|_{\rho})$$

we have

$$\mathscr{H}^{c\rho,g}_{c\delta}(X) \leqslant \sup_{0<t\leqslant\delta}\frac{g(ct)}{g(t)}\mathscr{H}^{\rho,g}_{\delta}(X).$$

As $\mathscr{H}^{\rho,g}(X)$ is finite positive by the condition (ii) and $\lim_{t\to 0}\frac{g(ct)}{g(t)}=0$ by the condition (i), it follows that $\mathscr{H}^{c\rho,g}(X)=0$ by letting $\delta\to 0$.

Similarly, note that, for each $\delta>0$, $\{U_i\}_{i=1}^{\infty}$ is a $c\delta$-cover of X with respect to ρ if and only if $\{U_i\}_{i=1}^{\infty}$ is a δ-cover of X with respect to $\frac{1}{c}\rho$, by

$$\sum_{i=1}^{\infty}g(|U_i|_{\frac{1}{c}\rho}) \geqslant \inf_{0<t\leqslant\delta}\frac{g(t)}{g(ct)}\sum_{i=1}^{\infty}g(|U_i|_{\rho})$$

we have

$$\mathscr{H}^{\frac{1}{c}\rho,g}_{\delta}(X) \geqslant \inf_{0<t\leqslant\delta}\frac{g(t)}{g(ct)}\mathscr{H}^{\rho,g}_{c\delta}(X).$$

By letting $\delta\to 0$, it follows that $\mathscr{H}^{\frac{1}{c}\rho,g}(X)=+\infty$.

Given a metric space (X,ρ) and a gauge g, $\mathscr{H}^{c\rho,g}(X)$, as a function of variable $c\in(0,\infty)$, is increasing. However we will show that even if the gauge satisfies the doubling condition and $0<\mathscr{H}^{\rho,g}(X)<\infty$, $\mathscr{H}^{c\rho,g}(X)$ may not be strictly increasing, in fact, we will construct a gauge function g and a metric space (X,ρ) such that for all $c\in\left[\frac{1}{2},1\right]$, $\mathscr{H}^{c\rho,g}(X)=\frac{1}{3}$.

Now let ρ be the Euclidean metric on the real line. Consider an iterated function system (IFS) on $[0,1]$:

$$\phi=(\varphi_1,\varphi_2,\varphi_3), \quad \varphi_1(x)=\frac{x}{5},$$
$$\varphi_2(x)=\frac{x}{5}+\frac{2}{5}, \quad \varphi_3(x)=\frac{x}{5}+\frac{4}{5}.$$

Let X be the self-similar set generated by the IFS ϕ, then $X=\bigcup_{1\leqslant i\leqslant 3}\varphi_i(X)$. It is known that X is a Cantor-type set with Hausdorff dimension $s=\log_5 3$ and s-dimensional Hausdorff measure $\mathscr{H}^s(X)=1$. For the details about IFS and self-similar set, we refer to Falconer[5].

Set

$$g(t)=\begin{cases}0 & \text{if } t=0,\\ 3^{-n} & \text{if } 5^{-n}\leqslant t<5^{-n+1},\ n\in\mathbb{N}.\end{cases}$$

Proposition 2. With the above notations, we have

$$\mathscr{H}^{\rho,g}(X)=\mathscr{H}^{\frac{1}{2}\rho,g}(X)=\frac{1}{3}.$$

Proof. By the definition of g, it is easily seen that g is a doubling gauge satisfying $\frac{1}{3}t^s\leqslant g(t)\leqslant t^s$ for $0\leqslant t<1$, thus $\frac{1}{3}\leqslant\mathscr{H}^{\rho,g}(X)\leqslant 1$. To prove the conclusion of the proposition, it suffices to show that $\mathscr{H}^{\rho,g}(X)\leqslant\frac{1}{3}$ and $\mathscr{H}^{\frac{1}{2}\rho,g}(X)\geqslant\frac{1}{3}$.

Let $n,j\in\mathbb{N}$. By the construction of X, we see that X consists of 3^n copies of itself of diameter 5^{-n}, and each copy can be covered by two intervals of which the lengths are respectively $5^{-n}-5^{-(n+j)}$ and $5^{-(n+j)}$, so X can be covered by $2\cdot 3^n$ intervals, half with length $5^{-n}-5^{-(n+j)}$ and the other half with length $5^{-(n+j)}$. Thus

$$\mathscr{H}^{\rho,g}_{5^{-n}}(X)\leqslant 3^n g(5^{-n}-5^{-(n+j)})+3^n g(5^{-(n+j)})$$
$$=3^n3^{-(n+1)}+3^n3^{-(n+j)}=\frac{1}{3}+\frac{1}{3^j},$$

which yields $\mathscr{H}^{\rho,g}(X)\leqslant\frac{1}{3}$.

On the other hand, let μ be the Borel measure on $\frac{1}{2}X\left(:=\left\{\frac{1}{2}x,x\in X\right\}\right)$ defined by $\mu(I_n)=3^{-n}$ for every copy I_n of level-n of $\frac{1}{2}X$, $n\in\mathbb{N}$. Suppose I is an interval with $0<|I|<1$. Let $n\in\mathbb{N}$ be the integer with $5^{-n}\leqslant|I|<5^{-n+1}$ and $\tilde{I}(\subseteq[0,1])$ an interval of length 5^{-n+1} such that $I\subseteq\tilde{I}$. By the construction of $\frac{1}{2}X$, every copy of level-$(n-1)$ of $\frac{1}{2}X$ has diameter $5^{-n+1}/2$, and any two adjacent copies of level-$(n-1)$ are separated by a gap of length $5^{-n+1}/2$. Thus $\tilde{I}$ intersects at most two copies of level-$(n-1)$ of $\frac{1}{2}X$, which leads to $\mu(\tilde{I})=3^{-n+1}$. Thus

$$\mu(I)\leqslant\mu(\tilde{I})=3g(|I|).$$

From the Frostman Lemma, we obtain $\mathscr{H}^{\frac{1}{2}\rho,g}(X)\geqslant\frac{1}{3}$.

References

1 Mattila, P. Geometry of Sets and Measures in Euclidean Spaces. Cambridge: Cambridge University Press, 1995.

2 Mauldin, R. D. et al. Scaling Hausdorff measures. Mathematika, 1989, 36: 325.

3 Csörnyei, C. et al. Scaling properties of Hausdorff and packing measures. Math. Ann., 2001, 319: 817.

4 Rogers, C. A. Hausdorff Measures. Cambridge: Cambridge University Press, 1970.

5 Falconer, K. J. Techniques in Fractal Geometry. New York: John Wiley and Sons, 1997.